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Analyticity of the semigroup associated with the fluid–rigid body problem and local existence of strong solutions ☆

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Abstract

In this paper, we study the linear operator associated with the fluid–rigid body problem. The operator was first introduced by T. Takahashi and M. Tucsnak (2004) [22]. For the general three-dimensional case, we prove that the corresponding semigroup is analytic on $L^{\frac{6}{5}}(\mathbb{R}^3) \cap L^p(\mathbb{R}^3)$ $(p \geqslant 2)$. In particular, when the solid is a ball in $\mathbb{R}^3$, the corresponding semigroup is analytic on $L^2(\mathbb{R}^3) \cap L^p(\mathbb{R}^3)$ $(p \geqslant 6)$. And for this case, a unique local strong solution to the fluid–rigid body problem is derived.

Keywords: Fluid–rigid body system; Semigroup; Navier–Stokes equations; Exterior domain

1. Introduction

Many physical phenomena involve interactions between moving structures and fluids. An interesting problem arising in fluid mechanics is the motion of a rigid body immersed in a viscous

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incompressible fluid. The motion of the fluid is governed by the classical Navier–Stokes equations with the non-slip boundary condition. The motion of the rigid body, consisting of a translation part and a rotation part, is ruled by the conservation laws of linear and angular momentum.

Suppose the region occupied by the homogeneous rigid body at time t is denoted by $O(t)$, and the domain occupied by the homogeneous fluid is $\Omega(t)=\mathbb{R}^3\setminus\overline{O(t)}$. Let $O(0)=O$, and $\Omega(0)=\Omega$. For the sake of simplicity, we assume that both the fluid and the solid are homogeneous with density 1. Then the system modeling the motion of the fluid and the rigid body is,

$$\begin{cases} \dfrac{\partial u}{\partial t}-\nu\Delta u+(u\cdot\nabla)u+\nabla p=0, \quad (x,t)\in\Omega(t)\times(0,T),\\ \operatorname{div}u=0, \quad (x,t)\in\Omega(t)\times[0,T),\\ u(x,t)=h'(t)+\omega(t)\times\big(x-h(t)\big), \quad (x,t)\in\partial\Omega(t)\times[0,T),\\ mh''(t)=-\displaystyle\int\limits_{\partial\Omega(t)}\big(\sigma(u,p)\vec{n}\big)\,d\Gamma, \quad t\in[0,T),\\ (J\omega)'(t)=-\displaystyle\int\limits_{\partial\Omega(t)}\big(x-h(t)\big)\times\sigma(u,p)\vec{n}\,d\Gamma, \quad t\in[0,T),\\ u(x,0)=a(x), \quad x\in\Omega,\\ h(0)=0\in\mathbb{R}^3, \qquad h'(0)=b\in\mathbb{R}^3, \qquad \omega(0)=c\in\mathbb{R}^3. \end{cases} \tag{1.1}$$

In the above system, $u=(u_1,u_2,u_3)$ and p denote the velocity field and the pressure of the fluid respectively. $\vec{n}(t)$ is the unit outward normal vector to $\partial\Omega(t)$. $h(t)$ and $\omega(t)$ denote the position of the center and the angular velocity of the solid at the time t respectively. m is the mass of the rigid body, i.e.,

$$m=\int\limits_{O(t)}dx=\int\limits_{O}dx,$$

$J(t)=(J_{kl}(t))$ is the moment of inertia related to the mass center of the rigid body, i.e.,

$$J_{kl}(t)=\int\limits_{O(t)}\Big[\big|x-h(t)\big|^2\delta_{kl}-\big(x-h(t)\big)_k\big(x-h(t)\big)_l\Big]\,dx.$$

Here δ_{kl} is the Kronecker symbol, and $\sigma(u,p)$ is the Cauchy stress tensor field,

$$\sigma(u,p)=-p\,Id+2\nu D(u),$$

where Id is the identity matrix and $D(u)$ is the deformation tensor,

$$D(u)=\frac{1}{2}\big[\nabla u+(\nabla u)^T\big].$$

There have been extensive research on the problem (1.1) in the last few years. The global existence of weak solutions to (1.1) has already been proved by [15] and [21]. When the fluid–rigid body system occupies a bounded domain, the existence of weak solutions has been studied by many mathematicians, see e.g. [4,5,9–11,20]. Furthermore, the collision between the solid and the domain's boundary has been investigated, see [12,13] and references therein.

However, only a few results are available on the existence and uniqueness of strong solutions. For the case that the rigid body is a disk in $\mathbb{R}^2$, T. Takahashi and M. Tucsnak [22] showed the existence and uniqueness of global strong solutions. Later, P. Cumsille and T. Takahashi [3] extended the global existence result to the general rigid body case in $\mathbb{R}^2$. For the three-dimensional case, they proved the local existence and uniqueness of strong solutions in $C[0,T; W^{1,2}(\mathbb{R}^3))$, see also [8] for the local existence of strong solutions. The research methods in [3,22] are totally different from that for the weak solutions.

When the fluid is inviscid, the Navier–Stokes equations in (1.1) will be replaced by the Euler equations and the boundary conditions change correspondingly. There also some results, see [18,19] and references therein.

Since the domain occupied by the fluid is varying with time and not a priori known, it's a free boundary problem. The problem can be transformed into an equivalent fixed boundary problem by moving along the center of the rigid body. More precisely, suppose that O is a ball in $\mathbb{R}^3$, let

$$\begin{gathered}
y = x - h(t), \qquad v(y,t) = u\big(y + h(t), t\big), \\
q(y,t) = p\big(y + h(t), t\big), \qquad l(t) = h'(t), \\
\sigma(v,q) = -q(y,t)Id + 2\nu D(v)(y,t), \\
J = (J_{kl}) = \Bigg(\int\limits_O \big[|y|^2\delta_{kl} - y_k y_l\big]\, dy \Bigg).
\end{gathered}$$

Then the problem (1.1) becomes

$$\begin{cases}
\dfrac{\partial v}{\partial t} - \nu\Delta v + (v\cdot\nabla)v - (l\cdot\nabla)v + \nabla q = 0, \quad (y,t) \in \Omega \times (0,T), \\
\operatorname{div} v = 0, \quad (y,t) \in \Omega \times [0,T), \\
v(y,t) = l(t) + \omega(t) \times y, \quad (y,t) \in \partial\Omega \times [0,T), \\
ml'(t) = -\displaystyle\int\limits_{\partial\Omega} \sigma(v,q)\vec{n}\, d\Gamma, \quad t \in [0,T), \\
J\omega'(t) = -\displaystyle\int\limits_{\partial\Omega} y \times \big[\sigma(v,q)\vec{n}\big]\, d\Gamma, \quad t \in [0,T), \\
v(y,0) = a(y), \quad y \in \Omega, \\
l(0) = b \in \mathbb{R}^3, \qquad \omega(0) = c \in \mathbb{R}^3.
\end{cases} \tag{1.2}$$

Remark 1.1. In this paper, we just study the strong solutions for the particular case that O is a ball. When the solid is of general shape, refer to [3] for the transformation.

To study the new system (1.2), the authors of [22] and [3] applied the method of semigroups. They extended v to a function defined on the whole space by letting $v(y,t) = l(t) + \omega(t) \times y$ in O and defined a new linear operator A_2.

$$D(A_2) = \{v \in W^{1,2}(\mathbb{R}^3)\colon \operatorname{div} v = 0 \text{ in } \mathbb{R}^3,\ D(v) = 0 \text{ in } O,\ v|_\Omega \in W^{2,2}(\Omega)\},$$
$$\mathcal{A}_2 v = \begin{cases} -\nu\Delta v & \text{in } \Omega, \\ \frac{2\nu}{m}\int_{\partial\Omega} D(v)\vec{n}\,d\Gamma + 2\nu J^{-1}[\int_{\partial\Omega} y \times D(v)\vec{n}\,d\Gamma] \times y & \text{in } O, \end{cases}$$

and

$$A_2 v = \mathbb{P}\mathcal{A}_2 v.$$

Here $\mathbb{P}$ is the orthogonal projector from $L^2(\mathbb{R}^3)$ onto its subspace H_1^2, where

$$H_1^2 = \{v \in L^2(\mathbb{R}^3)\colon \operatorname{div} v = 0 \text{ in } \mathbb{R}^3,\ D(v) = 0 \text{ in } O\}.$$

Omitting the nonlinear terms $(v \cdot \nabla)v$ and $(l \cdot \nabla)v$ in the first equation of (1.2), one can get the corresponding linearized system. Then A_2 is the linear operator associated with the linearized system, since [22] has proved that the linearized system equals to the following abstract equation in some sense,

$$\begin{cases} \partial_t v + A_2 v = 0, \\ v(y,0) = \begin{cases} a(y), & y \in \Omega, \\ b + c \times y, & y \in O. \end{cases} \end{cases}$$

In [3], it was proved that $-A_2$ is a dissipative operator on the Hilbert space H_1^2. With this result at hand, it is easy to verify that

$$\left\|(\sigma + i\tau + A_2)^{-1}\right\| \leqslant \frac{1}{|\tau|},$$

where the norm is the norm for an operator from H_1^2 to H_1^2 and $\sigma > 0$, $\tau \neq 0$. According to Theorem 3.2.7 in [1], actually it gives the following theorem.

Theorem 1.2. *The linear operator $-A_2$ is the infinitesimal generator of an analytic semigroup $\{e^{-tA_2}\}$ of operators on the space H_1^2.*

In this paper, we prove that the corresponding operator is also the generator of an analytic semigroup on $H_1^{\frac{6}{5}} \cap H_1^p$ $(p \geqslant 2)$. When the solid is a ball in $\mathbb{R}^3$, we can also prove its analyticity on the space $H_1^2 \cap H_1^p$ $(p \geqslant 6)$. For the definition of H_1^p, please refer to Section 2.

As an application, we use the analytic semigroup to study the local well-posedness of the fluid–rigid body system, when the rigid body is a ball in $\mathbb{R}^3$. The main idea is the Fujita–Kato approach. In fact, the local existence and uniqueness of strong solutions in $H_1^{\frac{6}{5}} \cap H_1^p$ $(p > 3)$ space was derived. Similar results hold in $H_1^2 \cap H_1^p$ $(p \geqslant 6)$. Note that the local strong solution derived in [8] and [3] required the initial data at least belongs to $W^{1,2}(\mathbb{R}^3)$, hence we extend the class of initial data. Their proofs rely strongly on the properties of Hilbert spaces, while our proof applies to more general setting. Furthermore, the properties of the linear operator derived here maybe useful for exploring more information about the original problem.

2. Main results and preliminaries

Before stating the main results in this paper, we introduce some function spaces and notations. Let O be a bounded, simply connected domain of C^2 in $\mathbb{R}^3$, and Ω be its exterior domain, $\Omega = \mathbb{R}^3 \setminus \overline{O}$. Without loss of generality, the center of O is supposed to be the origin, i.e.,

$$\int_O y\,dy = 0 \in \mathbb{R}^3.$$

Otherwise, one just needs to take a translation of the coordinates system. $\vec{n}$ denotes the outward unit normal to the boundary $\partial\Omega$. Let $m = \int_O dy$, and $J = (J_{kl})$,

$$J_{kl} = \int_O \big[|y|^2\delta_{kl} - y_k y_l\big]\,dy.$$

B_R is the ball in $\mathbb{R}^3$ centered at 0 and with the radius R. For any linear operator A, denote the domain of A by $D(A)$ and the range of A by $R(A)$. Denote the conjugate of a function f by $\overline{f}$. In the case of no ambiguity, we do not distinguish the notations of the vector-valued function spaces and the scalar function spaces strictly. $L^p(\Omega)$, $W^{k,p}(\Omega)$ are the usual Sobolev spaces defined on the domain Ω. And $L^p(\mathbb{R}^3)$, $W^{k,p}(\mathbb{R}^3)$ are the usual Sobolev spaces defined on $\mathbb{R}^3$. $C_{0,\sigma}^\infty(\Omega)$ consists of smooth functions defined on Ω with compact support and divergence free.

For $1 < p < \infty$, $D^{1,p}(\Omega) = \{u \in L^1_{loc}(\Omega)\colon\ \nabla u \in L^p(\Omega)\}$. In $D^{1,p}(\Omega)$, we introduce the seminorm

$$|u|_{1,p,\Omega} = \bigg[\int_\Omega |\nabla u|^p\,dy\bigg]^{1/p}.$$

If we identify the two functions $u_1, u_2 \in D^{1,p}(\Omega)$ whenever $|u_1 - u_2|_{1,p,\Omega} = 0$, i.e., u_1 and u_2 may differ by a constant, we denote the quotient space by $\dot{D}^{1,p}(\Omega)$. In the following text, without any confusion, we do not distinguish the elements in $D^{1,p}(\Omega)$ and $\dot{D}^{1,p}(\Omega)$ very strictly.

Let

$$\begin{gathered}
L^p_\sigma = \big\{u \in L^p\big(\mathbb{R}^3\big)\colon\ \operatorname{div} u = 0 \text{ in } \mathbb{R}^3\big\},\\
H^p_1 = \big\{u \in L^p\big(\mathbb{R}^3\big)\colon\ \operatorname{div} u = 0 \text{ in } \mathbb{R}^3,\ D(u) = 0 \text{ in } O\big\},\\
G^p_1 = \big\{u \in L^p\big(\mathbb{R}^3\big)\colon\ u = \nabla q_1,\ q_1 \in L^1_{loc}\big(\mathbb{R}^3\big)\big\},\\
G^p_2 = \left\{u \in L^p\big(\mathbb{R}^3\big)\ \middle|\ \begin{array}{l}\operatorname{div} u = 0 \text{ in } \mathbb{R}^3,\ u = \nabla q_2 \text{ in } \Omega,\ q_2 \in L^1_{loc}(\Omega),\ u = \phi \text{ in } O,\\ \int_O \phi\,dy = -\int_{\partial\Omega} q_2\vec{n}\,d\Gamma, \text{ and } \int_O \phi \times y\,dy = -\int_{\partial\Omega} q_2\vec{n} \times y\,d\Gamma\end{array}\right\}.
\end{gathered}$$

Concerning the functions in H_1^p, the following characterization holds.

Lemma 2.1. *Let $1 \leqslant p \leqslant \infty$, and $u \in H_1^p$. Then*

$$u(y) = l_u + \omega_u \times y \quad \text{in } O, \tag{2.3}$$

where

$$l_u = \frac{1}{m}\int_O u\,dy \quad \text{and} \quad \omega_u = -J^{-1}\int_O u \times y\,dy.$$

Here J^{-1} is the inverse of the matrix J.

The proof of Lemma 2.1 is very simple. One can get that by integrating (2.3) and (2.3) $\times\, y$ over the domain O.

Next, a theorem about the decomposition of $L^p(\mathbb{R}^3)$ is given, which is closely associated with the fluid–rigid body problem.

Theorem 2.2. *For $1 < p < \infty$,*

$$L^p(\mathbb{R}^3) = H_1^p \oplus G_1^p \oplus G_2^p.$$

Thus, for any $u \in L^p(\mathbb{R}^3)$, one has

$$u = v + \nabla q_1 + w \in H_1^p \oplus G_1^p \oplus G_2^p.$$

Set $v = \mathbb{P}_p u$, then $\mathbb{P}_p$ is the projection operator from $L^p(\mathbb{R}^3)$ onto H_1^p. In fact, $\mathbb{P}$ is a bounded operator.

Remark 2.3. When O is a disc in $\mathbb{R}^2$, the same decomposition of $L^2(\mathbb{R}^2)$ has been proved by M. Dashti and J.C. Robinson [6].

Remark 2.4. The same result holds for the two-dimensional case with some minor modification, which is

$$G_2^p = \left\{ u \in L^p(\mathbb{R}^3) \;\middle|\; \begin{array}{l} \operatorname{div} u = 0 \text{ in } \mathbb{R}^2,\ u = \nabla q_2 \text{ in } \Omega,\ q \in L^1_{loc}(\Omega),\ u = \phi \text{ in } O, \\ \int_O \phi\,dy = -\int_{\partial\Omega} q_2 \vec{n}\,d\Gamma, \text{ and } \int_O \phi \cdot y^\perp\,dy = -\int_{\partial\Omega} q_2 y^\perp \cdot \vec{n}\,d\Gamma \end{array} \right\}.$$

As indicated in the proof of Theorem 2.2, for every $u \in L^p(\mathbb{R}^3) \cap L^q(\mathbb{R}^3)$, $p \neq q$, $\mathbb{P}_p u = \mathbb{P}_q u$. Hence, we will omit the subindex of $\mathbb{P}_p$, and just write $\mathbb{P}$ instead in this paper.

Set

$$D(A_{\frac{6}{5}\cap p}) = \left\{ v \in W^{1,\frac{6}{5}}(\mathbb{R}^3) \cap W^{1,p}(\mathbb{R}^3) \;\middle|\; \begin{array}{l} \operatorname{div} v = 0 \text{ in } \mathbb{R}^3,\ D(v) = 0 \text{ in } O, \\ v|_\Omega \in W^{2,\frac{6}{5}}(\Omega) \cap W^{2,p}(\Omega) \end{array} \right\}. \tag{2.4}$$

For any $v \in D(A_{\frac{6}{5}\cap p})$, define

$$\mathcal{A}_{\frac{6}{5}\cap p}v = \begin{cases} -\nu\Delta v & \text{in } \Omega, \\ \frac{2\nu}{m}\int_{\partial\Omega} D(v)\vec{n}\,d\Gamma + 2\nu J^{-1}[\int_{\partial\Omega} y \times D(v)\vec{n}\,d\Gamma] \times y & \text{in } O, \end{cases} \tag{2.5}$$

and

$$A_{\frac{6}{5}\cap p}v = \mathbb{P}\mathcal{A}_{\frac{6}{5}\cap p}v. \tag{2.6}$$

Similarly, one can define the space $D(A_{2\cap p})$, the linear operators $\mathcal{A}_{2\cap p}$ and $A_{2\cap p}$, through replacing $\frac{6}{5}$ by 2 in (2.4), (2.5) and (2.6).

Now our main result reads as:

Theorem 2.5. *For any $2 \leqslant p < \infty$, the linear operator $-A_{\frac{6}{5}\cap p}$ is the infinitesimal generator of the analytic semigroup $\{e^{-tA_{\frac{6}{5}\cap p}}\}$ of operators on $H_1^{\frac{6}{5}} \cap H_1^p$. And for every $t > 0$, it holds that*

$$\left\| e^{-tA_{\frac{6}{5}\cap p}} \right\| \leqslant M_1, \qquad \left\| A^k_{\frac{6}{5}\cap p} e^{-tA_{\frac{6}{5}\cap p}} \right\| \leqslant \frac{M_1}{|t|^k}, \tag{2.7}$$

with $M_1 = M_1(p,\Omega) > 0$. Then it follows that for every $u \in H_1^{\frac{6}{5}} \cap H_1^p$,

$$\lim_{t\to+\infty} \left\| e^{-tA_{\frac{6}{5}\cap p}} u \right\|_{L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)} = 0.$$

The corresponding result for the classical Stokes operator $\tilde{A}_p$ over the domain Ω was proved in [2], which reads as:

Proposition 2.6. *Let $1 < p < \infty$, $0 < \theta < \frac{\pi}{2}$. Then for every $\lambda \in \mathbb{C}$ with $|\lambda| > 0$, $|\arg\lambda| \leqslant \frac{\pi}{2} + \theta$, the resolvent $(\lambda I + \tilde{A}_p)^{-1}$ of the operator $\tilde{A}_p$ exists and it holds*

$$\left\| (\lambda I + \tilde{A}_p)^{-1} \right\| \leqslant \frac{C}{|\lambda|} \quad \textit{for all } |\lambda| > 0, \qquad |\arg\lambda| \leqslant \frac{\pi}{2} + \theta,$$

where $C = C(p,\theta,\Omega) > 0$. And it follows that the semigroup $\{e^{-t\tilde{A}_p}\}$ is analytic for $t \in \mathbb{C}$, $t \neq 0$, and $|\arg t| < \theta$.

More concretely, taking into account the result in [17], Proposition 2.6 can be restated as follows:

Proposition 2.7. *Let $1 < p < \infty$, $0 < \theta < \frac{\pi}{2}$. Then for every $\lambda \in \mathbb{C}$ with $|\lambda| > 0$, $|\arg\lambda| \leqslant \frac{\pi}{2} + \theta$, and every $f \in L^p(\Omega)$, the system*

$$\begin{cases} \lambda u - \nu\Delta u + \nabla p = f & \textit{in } \Omega, \\ \operatorname{div} u = 0 & \textit{in } \Omega, \\ u = 0 & \textit{on } \partial\Omega, \end{cases}$$

has a unique solution $u \in W^{2,p}(\Omega)$ with the following estimates,

$$|\lambda| \cdot \|u\|_{L^p(\Omega)} \leqslant C(p,\Omega)\|f\|_{L^p(\Omega)}, \tag{2.8}$$

$$\left\|D^2 u\right\|_{L^p(\Omega)} + \|\nabla p\|_{L^p(\Omega)} \leqslant C(p,\Omega)\left[\|f\|_{L^p(\Omega)} + \|u\|_{L^p(\Omega)}\right], \tag{2.9}$$

where $C(p,\Omega)$ is some constant depending only on p and Ω.

Remark 2.8. Comparing Theorem 2.5 to Proposition 2.6, we would like to prove that $-A_p$ is analytic on H_1^p. We cannot achieve this now. However, when O is a ball in $\mathbb{R}^3$, a further result is derived.

Theorem 2.9. *Suppose O is a ball of radius 1 in $\mathbb{R}^3$. For any $6 \leqslant p < \infty$, the linear operator $-A_{2\cap p}$ is the infinitesimal generator of the analytic semigroup $\{e^{-tA_{2\cap p}}\}$ of operators on $H_1^2 \cap H_1^p$. And for every $t > 0$, it holds that*

$$\left\|e^{-tA_{2\cap p}}\right\| \leqslant M, \qquad \left\|A_{2\cap p}^k e^{-tA_{2\cap p}}\right\| \leqslant \frac{M}{|t|^k},$$

with $M = M(p,\Omega) > 0$. Then it follows that for every $u \in H_1^2 \cap H_1^p$,

$$\lim_{t\to+\infty}\left\|e^{-tA_{2\cap p}}u\right\|_{L^2(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)} = 0.$$

Remark 2.10. Theorem 2.5 and Theorem 2.9 are the key estimates for establishing the local strong solution in $H_1^{\frac{6}{5}} \cap H_1^p$ and $H_1^2 \cap H_1^p$ respectively. The assumption that the initial data of system (1.2) belongs to H_1^2 is necessary in some sense, otherwise one may not get the uniform bound of the velocity of the solid. Hence Theorem 2.9 seems better and more reasonable. However, whether the conclusion of Theorem 2.9 holds for $2 < p < 6$ is open.

Remark 2.11. Although there are some differences between the three-dimensional case and two-dimensional case, our proof of Theorem 2.9 also applies to the corresponding case in the two-dimensional space.

As an application of Theorem 2.5, the particular case that the solid is the unit ball in $\mathbb{R}^3$ is studied. We show that

Theorem 2.12. *Assume O is the unit ball in $\mathbb{R}^3$ and $p > 3$. Let the initial data*

$$v_0(y) = \begin{cases} a(y), & y \in \Omega, \\ b + c \times y, & y \in O. \end{cases}$$

Suppose $v_0 \in H_1^{\frac{6}{5}} \cap H_1^p$, then there exists a unique local strong solution $v \in C([0,T_0]; H_1^{\frac{6}{5}} \cap H_1^p)$ to the system (1.2), and v satisfies

$$t^{\frac{1}{2}} v(y,t) \in C\left([0,T_0]; W^{1,\frac{6}{5}}\left(\mathbb{R}^3\right) \cap W^{1,p}\left(\mathbb{R}^3\right)\right).$$

3. Proof of Theorem 2.2

Before the proof of Theorem 2.2, two lemmas are cited for later use.

Lemma 3.1 *(The Helmholtz–Weyl Decomposition of $L^p(\mathbb{R}^3)$). For every $1 < p < +\infty$, $L^p(\mathbb{R}^3) = L^p_\sigma \oplus G^p_1$. In other words, for any vector function $u \in L^p(\mathbb{R}^3)$, it can be uniquely decomposed as the sum*

$$u = w_1 + w_2, \quad w_1 \in G^p_1,\ w_2 \in L^p_\sigma,$$

and there exists a constant $C(p)$ depending only on p, such that

$$\|w_i\|_{L^p(\mathbb{R}^3)} \leqslant C(p)\|u\|_{L^p(\mathbb{R}^3)}, \quad \text{for } i = 1, 2.$$

When $p = 2$, $L^2_\sigma \perp G^2_1$.

Lemma 3.2. *Assume that $1 < p < +\infty$. For every vector function $u \in L^p(\Omega)$, the Neumann problem*

$$\begin{cases} \Delta q = \operatorname{div} u & \text{in } \Omega, \\ \nabla q \cdot \vec{n} = u \cdot \vec{n} & \text{on } \partial\Omega, \end{cases}$$

has a unique (up to a constant) solution $q \in D^{1,q}(\Omega)$ and

$$\|\nabla q\|_{L^p(\Omega)} \leqslant C(p, \Omega)\|u\|_{L^p(\Omega)},$$

with some positive constant $C(p, \Omega)$.

The proof of Lemma 3.1 and Lemma 3.2 can be found in [7].

Proof of Theorem 2.2. As said in Remark 2.3, Theorem 2.2 was proved in [6], when O is a disc in $\mathbb{R}^2$ and $p = 2$. In fact, if O is a general smooth domain of $\mathbb{R}^3$, the proof for the case $p = 2$ is exactly the same. So we omit it. When $p \neq 2$, first we show that for every vector function $u \in C^\infty_0(\mathbb{R}^3)$, u has a decomposition as $u = v + \nabla q_1 + w \in H^p_1 + G^p_1 + G^p_2$.

Since $u \in C^\infty_0(\mathbb{R}^3) \subset L^2(\mathbb{R}^3)$, there exists a unique decomposition of u in $L^2(\mathbb{R}^3)$,

$$u = \begin{cases} v + \nabla q_1 + \nabla q_2 & \text{in } \Omega, \\ l_v + \omega_v y^\perp + \nabla q_1 + \phi & \text{in } O. \end{cases} \tag{3.10}$$

Herein, ∇q_1 is obtained by the Helmohltz–Weyl decomposition and

$$l_v = \frac{1}{m}\left[\int_O (u - \nabla q_1 - \phi)\, dy\right] = \frac{1}{m}\left[\int_O (u - \nabla q_1)\, dy + \int_{\partial\Omega} q_2 \vec{n}\, d\Gamma\right], \tag{3.11}$$

$$\begin{aligned}\omega_v &= -J^{-1}\Big[\int_O (u - \nabla q_1 - \phi) \times y\, dy\Big]\\ &= -J^{-1}\Big[\int_O (u - \nabla q_1) \times y\, dy + \int_{\partial\Omega} q_2 \vec{n} \times y\, d\Gamma\Big]. \end{aligned} \tag{3.12}$$

Since $u \in L^2(\mathbb{R}^3) \cap L^p(\mathbb{R}^3)$, then

$$\nabla q_1 \in L^p(\mathbb{R}^3),$$

and

$$\|\nabla q_1\|_{L^p(\mathbb{R}^3)} \leqslant C(p)\|u\|_{L^p(\mathbb{R}^3)},$$

with some constant $C(p)$ depending only on p.

We shall prove that $q_2 \in D^{1,p}(\Omega)$ and there exists a constant C independent of u, such that

$$\|\nabla q_2\|_{L^p(\Omega)} \leqslant C\|u\|_{L^p(\mathbb{R}^3)}. \tag{3.13}$$

In fact, it suffices to show that

$$|l_v| + |\omega_v| \leqslant C\|u\|_{L^p(\mathbb{R}^3)}, \tag{3.14}$$

with some uniform constant C independent of u. This is due to the fact that q_2 solves the problem:

$$\begin{cases} \Delta q_2 = 0 \quad \text{in } \Omega, \\ \nabla q_2 \cdot \vec{n} = (u - \nabla q_1) \cdot \vec{n} - l_v \cdot \vec{n} - (\omega_v \times y) \cdot \vec{n} \quad \text{on } \partial\Omega. \end{cases} \tag{3.15}$$

Take $\varphi \in C_0^\infty(\mathbb{R})$, with

$$\varphi(x) = \begin{cases} 1 & \text{if } |x| \leqslant 1, \\ 0 & \text{if } |x| \geqslant 2. \end{cases}$$

Set $\varphi_R(y) = \varphi(|y|/R)$, with some R large enough such that $O \subset B_{\frac{R}{2}}$. Let

$$\tilde{u} = u - \nabla q_1 - \operatorname{curl}\Big[\frac{1}{2} l_v \times y \varphi_R(y) + \frac{1}{2}\omega_v |y|^2 \varphi_R(y)\Big].$$

It can be verified that

$$\begin{cases} \Delta q_2 = \operatorname{div} \tilde{u} \quad \text{in } \Omega, \\ \nabla q_2 \cdot \vec{n} = \tilde{u} \cdot \vec{n} \quad \text{on } \partial\Omega. \end{cases} \tag{3.16}$$

According to Lemma 3.2, the Neumann problem (3.16) has a unique solution $q_2 \in D^{1,p}(\Omega)$ and

$$\|\nabla q_2\|_{L^p(\Omega)} \leqslant C(p,\Omega)\|\tilde{u}\|_{L^p(\Omega)} \leqslant C(p,\Omega,R)\big[\|u\|_{L^p(\mathbb{R}^3)} + |l_v| + |\omega_v|\big]. \tag{3.17}$$

Suppose that (3.14) does not hold, then one can choose a sequence $\{u^n\} \subseteq C_0^\infty(\mathbb{R}^3)$, such that

$$\|u^n\|_{L^p(\mathbb{R}^3)} \to 0, \quad \text{and} \quad |l_v^n| + |\omega_v^n| = 1.$$

Here l_v^n and ω_v^n are associated with u^n as in (3.10),

$$u^n = \begin{cases} v^n + \nabla q_1^n + \nabla q_2^n & \text{in } \Omega, \\ l_v^n + \omega_v^n \times y + \nabla q_1^n + \varphi^n & \text{in } O. \end{cases}$$

Then

$$l_v^n \to l_v^* \quad \text{in } \mathbb{R}^3, \qquad \omega_v^n \to \omega_v^* \quad \text{in } \mathbb{R}^3,$$

and $|l_v^*| + |\omega_v^*| = 1$.

Following similar estimates to (3.17), $\{\|\nabla q_2^n\|_{L^p(\Omega)}\}$ is uniformly bounded. Since $D^{1,p}(\Omega)$ has the property of weak compactness, one can derive a subsequence (denoted also by $\{q_2^n\}$), and a function $q_2^* \in D^{1,p}(\Omega)$, such that

$$q_2^n \rightharpoonup q_2^* \quad \text{in } D^{1,p}(\Omega).$$

Moreover, since

$$\|\nabla q_1^n\|_{L^p(\mathbb{R}^3)} \leqslant C(p)\|u^n\|_{L^p(\mathbb{R}^3)},$$

then

$$\nabla q_1^n \to 0 \quad \text{in } L^p(\mathbb{R}^3).$$

Note that

$$l_v^n = \frac{1}{m}\left[\int_O (u^n - \nabla q_1^n)\,dy + \int_{\partial\Omega} q_2^n \vec{n}\,d\Gamma\right],$$

and

$$\omega_v^n = -J^{-1}\left[\int_O (u - \nabla q_1^n) \times y\,dy + \int_{\partial\Omega} q_2^n \vec{n} \times y\,d\Gamma\right].$$

As n goes to $+\infty$,

$$l_v^* = \frac{1}{m}\int_{\partial\Omega} q_2^* \vec{n}\,d\Gamma, \qquad \omega_v^* = -J^{-1}\int_{\partial\Omega} q_2^* \vec{n} \times y\,d\Gamma. \tag{3.18}$$

Since q_2^n solves the following problem:

$$\begin{cases} \Delta q_2^n = 0 & \text{in } \Omega, \\ \nabla q_2^n \cdot \vec{n} = (u^n - \nabla q_1^n) \cdot \vec{n} - l_v^n \cdot \vec{n} - (\omega_v^n \times y) \cdot \vec{n} & \text{on } \partial\Omega. \end{cases} \tag{3.19}$$

As n goes to $+\infty$, one gets

$$\begin{cases} \Delta q_2^* = 0 & \text{in } \Omega, \\ \nabla q_2^* \cdot \vec{n} = -l_v^* \cdot \vec{n} - (\omega_v^* \times y) \cdot \vec{n} & \text{on } \partial\Omega. \end{cases} \tag{3.20}$$

Let $\tilde{u}^* = \operatorname{curl}[\frac{1}{2} l_v^* \times y\varphi_R(y) + \frac{1}{2}\omega_v^* |y|^2 \varphi_R(y)] \in L^2(\Omega)$, then

$$\begin{cases} \Delta q_2^* = \operatorname{div} \tilde{u}^* & \text{in } \Omega, \\ \nabla q_2^* \cdot \vec{n} = \tilde{u}^* \cdot \vec{n} & \text{on } \partial\Omega. \end{cases}$$

Hence, by virtue of Lemma 3.2, $q_2^* \in D^{1,2}(\Omega)$. Taking q_2^* itself as the test function of Eq. (3.20) and combining (3.18), one gets that

$$\int_\Omega |\nabla q_2^*|^2 \, dy + \frac{1}{m} \left| \int_{\partial\Omega} q_2^* \vec{n} \, d\Gamma \right|^2 + \left(J^{-1} \int_{\partial\Omega} q_2^* \vec{n} \times y \, d\Gamma \right) \cdot \int_{\partial\Omega} q_2^* \vec{n} \times y \, d\Gamma = 0.$$

Hence

$$\nabla q_2^* = l_v^* = \omega_v^* = [0,0,0]^T.$$

It's a contradiction to the fact that $|V_v^*| + |\omega_v^*| = 1$! Therefore,

$$\|\nabla q_2\|_{L^p(\Omega)} \leqslant C\|u\|_{L^p(\mathbb{R}^3)}, \qquad |l_v| + |\omega_v| \leqslant C\|u\|_{L^p(\mathbb{R}^3)}.$$

Consequently,

$$\|v\|_{L^p(\mathbb{R}^3)} \leqslant C\|u\|_{L^p(\mathbb{R}^3)}, \qquad \|w\|_{L^p(\mathbb{R}^3)} \leqslant C\|u\|_{L^p(\mathbb{R}^3)}.$$

As $C_0^\infty(\mathbb{R}^3)$ is dense in $L^p(\mathbb{R}^3)$, one can construct a decomposition of arbitrary $u \in L^p(\mathbb{R}^3)$ by taking an approximate sequence $\{u^m\} \subseteq C_0^\infty(\mathbb{R}^3)$.

At last, we show that the decomposition is unique. If $u \in L^p(\mathbb{R}^3)$ has two decompositions, i.e.,

$$u = v + \nabla q_1 + w = \tilde{v} + \nabla\tilde{q}_1 + \tilde{w} \in H_1^p + G_1^p + G_2^p.$$

Concretely,

$$u = \begin{cases} v + \nabla q_1 + \nabla q_2 = \tilde{v} + \nabla\tilde{q}_1 + \nabla\tilde{q}_2 & \text{in } \Omega, \\ l_v + \omega_v \times y + \nabla q_1 + \phi = \tilde{l}_v + \tilde{\omega}_v \times y + \nabla\tilde{q}_1 + \tilde{\phi} & \text{in } O. \end{cases}$$

By the uniqueness of the Helmholtz–Weyl decomposition, $\nabla q_1 = \nabla\tilde{q}_1$.

Moreover,

$$l_v = \frac{1}{m}\Big[\int_{\partial\Omega} q_2\vec{n}\,d\Gamma + \int_O (u - \nabla q_1)\,dy\Big],$$

and

$$\omega_v = -J^{-1}\Big[\int_{\partial\Omega} q_2\vec{n}\times y\,d\Gamma + \int_O (u - \nabla q_1)\times y\,dy\Big].$$

The same equalities hold for $\tilde{q}_2, \tilde{l}_v$ and $\tilde{\omega}_v$ instead of q_2, l_v and ω_v.

Let $q = q_2 - \tilde{q}_2 \in D^{1,p}(\Omega)$, then q is the solution of the following system,

$$\begin{cases} \Delta q = 0 \quad \text{in } \Omega, \\ \nabla q\cdot\vec{n} + \dfrac{1}{m}\displaystyle\int_{\partial\Omega} q\vec{n}\,d\Gamma\cdot\vec{n} - J^{-1}\Big(\int_{\partial\Omega} q\vec{n}\times y\,d\Gamma\Big)\times y\cdot\vec{n} = 0 \quad \text{on } \partial\Omega. \end{cases}$$

By taking q itself as a test function, the system has been proved to have the only trivial solution, i.e., $\nabla q \equiv 0$. Then

$$\nabla q_2 = \nabla\tilde{q}_2 \quad \text{in } \Omega, \quad \text{and} \quad l_v = \tilde{V}_v, \qquad \omega_v = \tilde{\omega}_v.$$

The decomposition is unique. It completes the proof of Theorem 2.2. □

4. Proof of Theorem 2.5

Proof of Theorem 2.5. Choose some θ_0 such that $0 < \theta_0 < \frac{\pi}{2}$. Let $\Sigma_0 = \{\lambda \in \mathbb{C}\colon |\arg\lambda| \geqslant \theta_0, |\lambda| \neq 0\}$. In order to prove that $-A_{\frac{6}{5}\cap p}$ is the infinitesimal generator of an analytic semigroup $\{e^{-tA_{\frac{6}{5}\cap p}}\}$ of operators on $H_1^{\frac{6}{5}} \cap H_1^p$, it suffices to show that $\Sigma_0 \subseteq \rho(A_{\frac{6}{5}\cap p})$, and for any $\lambda \in \Sigma_0$ and $f \in H_1^{\frac{6}{5}} \cap H_1^p$, there exists some constant $C = C(\Sigma_0, p, \Omega)$ such that

$$\big\|(\lambda I - A_{\frac{6}{5}\cap p})^{-1} f\big\|_{L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)} \leqslant C|\lambda|^{-1}\|f\|_{L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)}.$$

It follows from Theorem 1.2 (or the fact that $-A_2$ is negative), that $\Sigma_0 \subseteq \rho(A_2)$. Thus, for every $\lambda \in \Sigma_0$ and every $f \in H_1^2$, there exists a function $u \in D(A_2)$, such that

$$(\lambda I - A_2)u = f. \tag{4.21}$$

Suppose $f \in H_1^{\frac{6}{5}} \cap H_1^p$, we shall prove that the solution $u \in D(A_{\frac{6}{5}\cap p})$, and

$$\|u\|_{L^{\frac{6}{5}}(\mathbb{R}^3)} + \|u\|_{L^p(\mathbb{R}^3)} \leqslant C(\Sigma_0, p, \Omega)|\lambda|^{-1}\big[\|f\|_{L^{\frac{6}{5}}(\mathbb{R}^3)} + \|f\|_{L^p(\mathbb{R}^3)}\big],$$

with some constant $C(\Sigma_0, p, \Omega)$ independent of f.

Taking the inner product of (4.21) with u in $L^2(\mathbb{R}^3)$, then

$$\langle \lambda u - A_2 u, u\rangle = \langle f, u\rangle. \tag{4.22}$$

For the term $\langle A_2 u, u\rangle$,

$$\begin{aligned}
\langle A_2 u, u\rangle &= \int_\Omega (-\nu\Delta u)\cdot \bar{u}\,dy + \int_O \frac{2\nu}{m}\int_{\partial O} D(u)\vec{n}\,d\Gamma \cdot \bar{u}\,dy \\
&\quad + \int_O \left[\left(2\nu J^{-1}\int_{\partial O} y\times D(u)\vec{n}\,d\Gamma\right)\times y\right]\cdot \bar{u}\,dy \\
&= \int_\Omega -2\nu\,\mathrm{div}\big(D(u)\big)\cdot \bar{u}\,dy + \frac{2\nu}{m}\int_{\partial O} D(u)\vec{n}\,d\Gamma\cdot (m\overline{V_u}) \\
&\quad + \left(2\nu J^{-1}\int_{\partial O} y\times D(u)\vec{n}\,d\Gamma\right)\cdot \int_O y\times(\overline{\omega_u}\times y)\,dy \\
&= 2\nu\int_\Omega \big|D(u)\big|^2\,dy - 2\nu\int_{\partial O} D(u)\bar{u}\cdot\vec{n}\,d\Gamma + 2\nu\int_{\partial O} D(u)\vec{n}\,d\Gamma\cdot\overline{V_u} \\
&\quad + 2\nu\int_{\partial O} y\times D(u)\vec{n}\,d\Gamma\cdot J^{-1}\int_O y\times(\overline{\omega_u}\times y)\,dy
\end{aligned}$$

where the second equality is the result of $\int_O y\,dy = 0$, and the third equality is due to the fact that J^{-1} is a symmetric matrix.

Since

$$\int_O y\times(\overline{\omega_u}\times y)\,dy = \int_O \overline{\omega_u}|y|^2 - y(y\cdot\overline{\omega_u})\,dy = J\overline{\omega_u},$$

and $D(u)$ is symmetric, then

$$\begin{aligned}
\langle A_2 u, u\rangle &= 2\nu\int_\Omega \big|D(u)\big|^2\,dy - 2\nu\int_{\partial O} D(u)\bar{u}\cdot\vec{n}\,d\Gamma \\
&\quad + 2\nu\int_{\partial O} D(u)\vec{n}\,d\Gamma\cdot\overline{V_u} + 2\nu\int_{\partial O} y\times D(u)\vec{n}\,d\Gamma\cdot\overline{\omega_u} \\
&= 2\nu\int_\Omega \big|D(u)\big|^2\,dy - 2\nu\int_{\partial O} D(u)\bar{u}\cdot\vec{n}\,d\Gamma \\
&\quad + 2\nu\int_{\partial O} D(u)\vec{n}\,d\Gamma\cdot\overline{V_u} + 2\nu\int_{\partial O} D(u)\vec{n}\cdot(\overline{\omega_u}\times y)\,d\Gamma
\end{aligned}$$

$$= 2\nu \int\limits_{\mathbb{R}^3} \left|D(u)\right|^2 dy$$

$$= \nu \int\limits_{\mathbb{R}^3} |\nabla u|^2 dy,$$

where the last equality is given by the fact $\operatorname{div} u = 0$ in $\mathbb{R}^3$.

Hence, (4.22) tells that

$$Re\,\lambda \|u\|^2_{L^2(\mathbb{R}^3)} - \nu \|\nabla u\|^2_{L^2(\mathbb{R}^3)} = Re\langle f, u\rangle, \tag{4.23}$$

$$Im\,\lambda \|u\|^2_{L^2(\mathbb{R}^3)} = Im\langle f, u\rangle. \tag{4.24}$$

If $Re\,\lambda \leqslant 0$,

$$\left|Re\,\lambda \|u\|_{L^2(\mathbb{R}^3)}\right| \leqslant \left|Re\langle f, u\rangle\right|. \tag{4.25}$$

(4.24) and (4.25) give that

$$|\lambda| \cdot \|u\|^2_{L^2(\mathbb{R}^3)} \leqslant \left|\langle f, u\rangle\right| \leqslant \|f\|_{L^2(\mathbb{R}^3)} \|u\|_{L^2(\mathbb{R}^3)},$$

which implies that

$$\|u\|_{L^2(\mathbb{R}^3)} \leqslant |\lambda|^{-1} \|f\|_{L^2(\mathbb{R}^3)}. \tag{4.26}$$

On the other hand,

$$\nu \|\nabla u\|^2_{L^2(\mathbb{R}^3)} \leqslant -Re\langle f, u\rangle \leqslant \|f\|_{L^{\frac{6}{5}}(\mathbb{R}^3)} \cdot \|u\|_{L^6(\mathbb{R}^3)} \leqslant C\|f\|_{L^{\frac{6}{5}}(\mathbb{R}^3)} \cdot \|\nabla u\|_{L^2(\mathbb{R}^3)},$$

which implies that

$$\|\nabla u\|_{L^2(\mathbb{R}^3)} \leqslant C\|f\|_{L^2(\mathbb{R}^3)}. \tag{4.27}$$

While for the case that $Re\,\lambda > 0$, $Re\,\lambda \leqslant \cot\theta_0 \cdot |Im\,\lambda|$. From (4.24), one gets that

$$\|u\|^2_{L^2(\mathbb{R}^3)} \leqslant \frac{1}{|Im\,\lambda|} \|f\|_{L^2(\mathbb{R}^3)} \|u\|_{L^2(\mathbb{R}^3)} \leqslant \frac{C}{|\lambda|} \|f\|_{L^2(\mathbb{R}^3)} \|u\|_{L^2(\mathbb{R}^3)},$$

which implies that

$$\|u\|_{L^2(\mathbb{R}^3)} \leqslant C|\lambda|^{-1} \|f\|_{L^2(\mathbb{R}^3)}. \tag{4.28}$$

On the other hand,

$$\begin{aligned}\nu\|\nabla u\|_{L^2(\mathbb{R}^3)}^2 &= Re\,\lambda\|u\|_{L^2(\mathbb{R}^3)}^2 - Re\langle f, u\rangle \\ &\leqslant C|Im\,\lambda|\|u\|_{L^2(\mathbb{R}^3)}^2 - Re\langle f, u\rangle \\ &= C\big|Im\langle f, u\rangle\big| - Re\langle f, u\rangle \\ &\leqslant C\|f\|_{L^{\frac{6}{5}}(\mathbb{R}^3)} \cdot \|u\|_{L^6(\mathbb{R}^3)} \\ &\leqslant C\|f\|_{L^{\frac{6}{5}}(\mathbb{R}^3)} \cdot \|\nabla u\|_{L^2(\mathbb{R}^3)},\end{aligned}$$

which implies that

$$\|\nabla u\|_{L^2(\mathbb{R}^3)} \leqslant C\|f\|_{L^{\frac{6}{5}}(\mathbb{R}^3)}. \tag{4.29}$$

Hence, for both cases one has

$$\|u\|_{L^2(\mathbb{R}^3)} \leqslant C|\lambda|^{-1}\|f\|_{L^2(\mathbb{R}^3)}, \tag{4.30}$$

and

$$\|u\|_{L^6(\mathbb{R}^3)} \leqslant C\|\nabla u\|_{L^2(\mathbb{R}^3)} \leqslant C\|f\|_{L^{\frac{6}{5}}(\mathbb{R}^3)}. \tag{4.31}$$

Furthermore,

$$|V_u| = \frac{1}{m}\left|\int\limits_O u\,dy\right| \leqslant C\|u\|_{L^2(O)} \leqslant C|\lambda|^{-1}\|f\|_{L^2(\mathbb{R}^3)}, \tag{4.32}$$

$$|\omega_u| = \left|J^{-1}\int\limits_O u \times y\,dy\right| \leqslant C\|u\|_{L^2(O)} \leqslant C|\lambda|^{-1}\|f\|_{L^{\frac{6}{5}}(\mathbb{R}^3)}. \tag{4.33}$$

And

$$|V_u| \leqslant C\|u\|_{L^6(\mathcal{O})} \leqslant C\|f\|_{L^{\frac{6}{5}}(\mathbb{R}^3)}, \tag{4.34}$$

$$|\omega_u| \leqslant C\|u\|_{L^6(\mathcal{O})} \leqslant C\|f\|_{L^{\frac{6}{5}}(\mathbb{R}^3)}. \tag{4.35}$$

As the relationship between u and f in (4.21), it was shown in [22] that there exists some $p \in \mathcal{D}'$ such that (u, p) satisfies the Stokes type system:

$$\begin{cases}\lambda u + \nu\Delta u + \nabla p = f \quad \text{in } \Omega, \\ \operatorname{div} u = 0 \quad \text{in } \Omega, \\ u(y) = V_u + \omega_u \times y \quad \text{on } \partial\Omega.\end{cases}$$

Take $\psi \in C_0^\infty(\mathbb{R})$, with

$$\psi(x) = \begin{cases} 1 & \text{if } |x| \leqslant 1, \\ 0 & \text{if } |x| \geqslant 2. \end{cases}$$

Set $\psi_R(y) = \psi(|y|/R)$, with R large enough such that $O \subseteq B_{R/2}(0)$. Let

$$\tilde{u} = \operatorname{curl}\left[\frac{1}{2} V_u \times y \psi_R(y)\right] - \operatorname{curl}\left[\frac{1}{2}\omega_u |y|^2 \psi_R(y)\right],$$

and $w = u - \tilde{u}$. It's easy to verify that

$$\begin{cases} \lambda w + \nu \Delta w + \nabla p = f - \lambda \tilde{u} - \nu \Delta \tilde{u} & \text{in } \Omega, \\ \operatorname{div} w = 0 & \text{in } \Omega, \\ w(y) = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.36}$$

According to Proposition 2.7, one has

$$|\lambda| \|w\|_{L^{\frac{6}{5}}(\Omega)} \leqslant C(\Omega)\big[\|f\|_{L^{\frac{6}{5}}(\Omega)} + |\lambda| \cdot \big(|V_u| + |\omega_u|\big) + |V_u| + |\omega_u|\big],$$

and

$$|\lambda| \|w\|_{L^p(\Omega)} \leqslant C(p, \Omega)\big[\|f\|_{L^p(\Omega)} + |\lambda| \cdot \big(|V_u| + |\omega_u|\big) + |V_u| + |\omega_u|\big].$$

By the estimates (4.32), (4.33), (4.34) and (4.35),

$$\begin{aligned} |\lambda| \|w\|_{L^{\frac{6}{5}}(\Omega)} + |\lambda| \|w\|_{L^p(\Omega)} &\leqslant C\big[\|f\|_{L^{\frac{6}{5}}(\Omega)} + \|f\|_{L^p(\Omega)} + \|f\|_{L^2(\mathbb{R}^3)} + \|f\|_{L^{\frac{6}{5}}(\mathbb{R}^3)}\big] \\ &\leqslant C(\Sigma_0, p, \Omega)\big[\|f\|_{L^{\frac{6}{5}}(\mathbb{R}^3)} + \|f\|_{L^p(\mathbb{R}^3)}\big]. \end{aligned}$$

Consequently,

$$\begin{aligned} |\lambda| \|u\|_{L^{\frac{6}{5}}(\mathbb{R}^3)} + |\lambda| \|u\|_{L^p(\mathbb{R}^3)} &\leqslant |\lambda||V_u| + |\lambda||\omega_u| + |\lambda| \|w\|_{L^{\frac{6}{5}}(\Omega)} + |\lambda| \|w\|_{L^p(\Omega)} \\ &\leqslant C(\Sigma_0, p, \Omega)\big[\|f\|_{L^{\frac{6}{5}}(\mathbb{R}^3)} + \|f\|_{L^p(\mathbb{R}^3)}\big]. \end{aligned} \tag{4.37}$$

It follows that $-A_{\frac{6}{5}\cap p}$ is the infinitesimal generator of an analytic semigroup $\{e^{-tA_{\frac{6}{5}\cap p}}\}$ of operators on $H_1^{\frac{6}{5}} \cap H_1^p$, which completes the proof of Theorem 2.5. $\square$

5. Proof of Theorem 2.9

In this section, O is the unit ball in $\mathbb{R}^3$. The main difference between the proof of Theorem 2.5 and that of Theorem 2.9 is the choice of the function $\tilde{u}$.

Proof of Theorem 2.9. As before, let θ_0, Σ_0 be the same as in the proof of Theorem 2.5. For any $\lambda \in \Sigma_0$ and any $f \in H_1^2$, since $\Sigma_0 \subseteq \rho(A_2)$, there exists some function $u \in D(A_2)$, such that

$$(\lambda I - A_2)u = f, \tag{5.38}$$

and

$$\|u\|_{L^2(\mathbb{R}^3)} \leqslant C(\Sigma_0, \Omega)|\lambda|^{-1}\|f\|_{L^2(\mathbb{R}^3)}.$$

Suppose that $f \in H_1^2 \cap H_1^p$, we shall prove that $u \in D(A_{2\cap p})$, and

$$\|u\|_{L^2(\mathbb{R}^3)} + \|u\|_{L^p(\mathbb{R}^3)} \leqslant C(\Sigma_0, p, \Omega)|\lambda|^{-1}\big[\|f\|_{L^2(\mathbb{R}^3)} + \|f\|_{L^p(\mathbb{R}^3)}\big]. \tag{5.39}$$

Suppose that $u = V_u + \omega_u \times y$ in O. Consequently,

$$|V_u| \leqslant C\|u\|_{L^2(O)} \leqslant C|\lambda|^{-1}\|f\|_{L^2(\mathbb{R}^3)}, \tag{5.40}$$

$$|\omega_u| \leqslant C\|u\|_{L^2(O)} \leqslant C|\lambda|^{-1}\|f\|_{L^2(\mathbb{R}^3)}. \tag{5.41}$$

In fact, (5.38) implies that

$$Re\,\lambda\|u\|^2_{L^2(\mathbb{R}^3)} - \nu\|\nabla u\|^2_{L^2(\mathbb{R}^3)} = Re\langle f, u\rangle,$$
$$Im\,\lambda\|u\|^2_{L^2(\mathbb{R}^3)} = Im\langle f, u\rangle.$$

If $Re\,\lambda \leqslant 0$, then by the Hölder's and Sobolev's inequalities,

$$\nu\|\nabla u\|^2_{L^2(\mathbb{R}^3)} \leqslant \big|Re\langle f, u\rangle\big| \leqslant C\|f\|_{L^2(\mathbb{R}^3)} \cdot \|u\|_{L^2(\mathbb{R}^3)} \leqslant C|\lambda|^{-1}\|f\|^2_{L^2(\mathbb{R}^3)}.$$

Similarly, for $Re\,\lambda > 0$, one has

$$\begin{aligned}
\nu\|\nabla u\|^2_{L^2(\mathbb{R}^3)} &= Re\,\lambda\|u\|^2_{L^2(\mathbb{R}^3)} - Re\langle f, u\rangle \\
&\leqslant C|Im\,\lambda|\|u\|^2_{L^2(\mathbb{R}^3)} - Re\langle f, u\rangle \\
&= C\big|Im\langle f, u\rangle\big| - Re\langle f, u\rangle \\
&\leqslant C\|f\|_{L^2(\mathbb{R}^3)} \cdot \|u\|_{L^2(\mathbb{R}^3)} \\
&\leqslant C|\lambda|^{-1}\|f\|^2_{L^2(\mathbb{R}^3)},
\end{aligned}$$

where we used the fact that for every $\lambda \in \Sigma_0$, $Re\,\lambda \leqslant C(\Sigma_0)|Im\,\lambda|$.

Hence, for both cases one has $\|\nabla u\|_{L^2(\mathbb{R}^3)} \leqslant C|\lambda|^{-\frac{1}{2}}\|f\|_{L^2(\mathbb{R}^3)}$.

According to the Sobolev imbedding theorem,

$$\|u\|_{L^6(\mathbb{R}^3)} \leqslant C\|\nabla u\|_{L^2(\mathbb{R}^3)} \leqslant C|\lambda|^{-\frac{1}{2}}\|f\|_{L^2(\mathbb{R}^3)}, \tag{5.42}$$

and it follows that

$$|V_u| \leqslant C\|u\|_{L^6(O)} \leqslant C|\lambda|^{-\frac{1}{2}}\|f\|_{L^2(\mathbb{R}^3)}, \tag{5.43}$$

$$|\omega_u| \leqslant C\|u\|_{L^6(O)} \leqslant C|\lambda|^{-\frac{1}{2}}\|f\|_{L^2(\mathbb{R}^3)}. \tag{5.44}$$

Then we consider two separate cases: $|\lambda| < \frac{1}{2}$ and $|\lambda| \geqslant \frac{1}{2}$. When $|\lambda| < \frac{1}{2}$, set

$$\tilde{u}(y) = \operatorname{curl}\left[\frac{1}{2}V_u \times y\psi_{\mu_1}(y)\right] + \operatorname{curl}\left[\omega_u|y|^{-1}\psi_{\mu_2}(y)\right],$$

with some constants $\mu_1, \mu_2 > 1$ to be determined and ψ_R being defined in the proof of Theorem 2.5.

Let $w = u - \tilde{u}$, then w satisfies the following problem,

$$\begin{cases} \lambda w + \nu\Delta w + \nabla p = f - \lambda\tilde{u} - \nu\Delta\tilde{u} & \text{in } \Omega, \\ \operatorname{div} w = 0 & \text{in } \Omega, \\ w(y) = 0 & \text{on } \partial\Omega. \end{cases}$$

It follows from Proposition 2.7 and the estimates (5.43), (5.44), that

$$\begin{aligned} |\lambda|\|w\|_{L^p(\Omega)} &\leqslant C\left[\|f\|_{L^p(\Omega)} + |\lambda|\|\tilde{u}\|_{L^p(\Omega)} + \|\Delta\tilde{u}\|_{L^p(\Omega)}\right] \\ &\leqslant C\|f\|_{L^p(\Omega)} + C\left[|\lambda|^{\frac{1}{2}}\mu_1^{\frac{3}{p}} + |\lambda|^{\frac{1}{2}}\mu_2^{\frac{3}{p}} + |\lambda|^{\frac{1}{2}}\mu_2^{-1+\frac{3}{p}}\right] \cdot \|f\|_{L^2(\mathbb{R}^3)} \\ &\quad + C\left[|\lambda|^{-\frac{1}{2}}\mu_1^{-2+\frac{3}{p}} + |\lambda|^{-\frac{1}{2}}\mu_2^{-1+\frac{3}{p}} + |\lambda|^{-\frac{1}{2}}\mu_2^{-3+\frac{3}{p}}\right] \cdot \|f\|_{L^2(\mathbb{R}^3)}. \end{aligned} \tag{5.45}$$

Setting $\mu_1 = |\lambda|^{-\frac{1}{2}}$, and $\mu_2 = |\lambda|^{-1}$, then one gets

$$|\lambda|\|w\|_{L^p(\Omega)} \leqslant C\|f\|_{L^p(\Omega)} + C\left[|\lambda|^{\frac{1}{2}-\frac{3}{2p}} + |\lambda|^{\frac{1}{2}-\frac{3}{p}}\right]\|f\|_{L^2(\mathbb{R}^3)}.$$

Since $p \geqslant 6$ and $|\lambda| \leqslant \frac{1}{2}$, combining the estimates (5.40), (5.41) and (5.45), one gets

$$\begin{aligned} |\lambda|\|u\|_{L^p(\mathbb{R}^3)} &\leqslant |\lambda|\|w\|_{L^p(\Omega)} + |\lambda|\|\tilde{u}\|_{L^p(\Omega)} + C|\lambda|\left[|V_u| + |\omega_u|\right] \\ &\leqslant C\left[\|f\|_{L^2(\mathbb{R}^3)} + \|f\|_{L^p(\mathbb{R}^3)}\right]. \end{aligned} \tag{5.46}$$

On the other hand, when $|\lambda| \geqslant \frac{1}{2}$, let

$$\tilde{u} = \operatorname{curl}\left[\frac{1}{2}V_u \times y\psi_1(y)\right] - \operatorname{curl}\left[\frac{1}{2}\omega_u|y|^2\psi_1(y)\right],$$

and $w = u - \tilde{u}$. Similar to (5.45), one has

$$|\lambda|\|w\|_{L^p(\Omega)} \leqslant C\big[\|f\|_{L^p(\Omega)} + |\lambda| \cdot \big(|V_u| + |\omega_u|\big) + |V_u| + |\omega_u|\big] \\ \leqslant C\big[\|f\|_{L^p(\mathbb{R}^3)} + \|f\|_{L^2(\mathbb{R}^3)}\big]. \tag{5.47}$$

From the estimates (5.40), (5.41) and (5.47), one can conclude that

$$|\lambda|\|u\|_{L^p(\mathbb{R}^3)} \leqslant |\lambda|\big[\|w\|_{L^p(\Omega)} + \|v\|_{L^p(\Omega)}\big] + C|\lambda|\big[|V_u| + |\omega_u|\big] \\ \leqslant C\big[\|f\|_{L^p(\mathbb{R}^3)} + \|f\|_{L^2(\mathbb{R}^3)}\big]. \tag{5.48}$$

Therefore, (5.39) holds, which completes the proof. □

Remark 5.1. In the case of two-dimensional motion, we just need to take the function $\tilde{u}(y)$ as $\nabla^{\perp}[V_u \cdot y^{\perp}\psi_{|\lambda|^{-1/2}}(y) - \omega_u \ln|y| \cdot \psi_{|\lambda|^{-1}}(y)]$.

6. L^q–L^r estimates

In this section, we give some L^q–L^r estimates associated with the semigroup $\{e^{-tA_{\frac{6}{5}\cap p}}\}$, which will be the key estimates for the proof of local well-posedness. In Section 6 and Section 7, for simplicity we will write A instead of $A_{\frac{6}{5}\cap p}$.

Proposition 6.1. *Assume that* $2 \leqslant p < \infty$*, and* q *satisfies that*

$$\begin{cases} q \in [p, \infty] & \text{if } 2 \leqslant p < 3, \\ q \in [p, \infty) & \text{if } p = 3, \\ q \in [p, \infty] & \text{if } p > 3. \end{cases}$$

Then there exist some positive constants $C_1(\Omega, p)$ *and* $C_2(\Omega, p, q)$ *such that, for any* $u_0 \in H_1^{\frac{6}{5}} \cap H_1^p$ *and* $t > 0$*,*

$$\big\|\nabla e^{-tA}u_0\big\|_{L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)} \leqslant C_1(\Omega, p)\big(1 + t^{-\frac{1}{2}}\big)\|u_0\|_{L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)}, \tag{6.49}$$

and

$$\big\|e^{-tA}u_0\big\|_{L^q(\mathbb{R}^3)} \leqslant C_2(\Omega, p, q)\big(1 + t^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{q})}\big)\|u_0\|_{L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)}. \tag{6.50}$$

Proof of Proposition 6.1. First, for any $u \in D(A)$, we derive an estimate of $\|\nabla u\|_{L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)}$ in terms of $\|u\|_{L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)}$ and $\|Au\|_{L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)}$. Suppose that

$$-u - Au = f,$$

and

$$u(y) = V_u + \omega_u \times y, \quad \text{in } O.$$

As in Section 4, there exists some $p \in \mathcal{D}'$ such that (u, p) satisfies the following system,

$$\begin{cases} -u + \nu\Delta u + \nabla p = f & \text{in } \Omega, \\ \operatorname{div} u = 0 & \text{in } \Omega, \\ u(y) = V_u + \omega_u \times y & \text{on } \partial\Omega. \end{cases}$$

Following the proof of Theorem 2.5, we choose some R large enough such that $O \subseteq B_{\frac{R}{2}}(0)$. Let

$$\tilde{u} = \operatorname{curl}\left[\frac{1}{2} V_u \times y \psi_R(y)\right] - \operatorname{curl}\left[\frac{1}{2}\omega_u |y|^2 \psi_R(y)\right],$$

and $w = u - \tilde{u}$. Then w is the solution to the problem

$$\begin{cases} -w + \nu\Delta w + \nabla p = f + \tilde{u} - \nu\Delta\tilde{u} & \text{in } \Omega, \\ \operatorname{div} w = 0 & \text{in } \Omega, \\ w(y) = 0 & \text{on } \partial\Omega. \end{cases}$$

By virtue of Proposition 2.7, one has

$$\begin{aligned} \|w\|_{W^{2,\frac{6}{5}}(\Omega)\cap W^{2,p}(\Omega)} &\leqslant C\big[\|f\|_{L^{\frac{6}{5}}(\Omega)\cap L^p(\Omega)} + \|\tilde{u}\|_{W^{2,\frac{6}{5}}(\Omega)\cap W^{2,p}(\Omega)} + \|w\|_{L^{\frac{6}{5}}(\Omega)\cap L^p(\Omega)}\big] \\ &\leqslant C\big[\|f\|_{L^{\frac{6}{5}}(\Omega)\cap L^p(\Omega)} + |V_u| + |\omega_u| + \|u\|_{L^{\frac{6}{5}}(\Omega)\cap L^p(\Omega)}\big] \\ &\leqslant C\big[\|f\|_{L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)} + \|u\|_{L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)}\big] \\ &\leqslant C\|f\|_{L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)}, \end{aligned}$$

where the last inequality comes from Theorem 2.5 by letting $\lambda = -1$.

Then

$$\begin{aligned} \|u\|_{W^{2,\frac{6}{5}}(\Omega)\cap W^{2,p}(\Omega)} &\leqslant \|w\|_{W^{2,\frac{6}{5}}(\Omega)\cap W^{2,p}(\Omega)} + \|\tilde{u}\|_{W^{2,\frac{6}{5}}(\Omega)\cap W^{2,p}(\Omega)} \\ &\leqslant C\big[\|f\|_{L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)} + |V_u| + |\omega_u|\big] \\ &\leqslant C\big[\|f\|_{L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)} + \|u\|_{L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)}\big] \\ &\leqslant C\|f\|_{L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)}. \end{aligned}$$

By the interpolation inequality,

$$
\begin{aligned}
\|\nabla u\|_{L^{\frac{6}{5}}(\Omega)\cap L^p(\Omega)} &\leqslant C\|u\|^{\frac{1}{2}}_{L^{\frac{6}{5}}(\Omega)\cap L^p(\Omega)} \cdot \|u\|^{\frac{1}{2}}_{W^{2,\frac{6}{5}}(\Omega)\cap W^{2,p}(\Omega)} \\
&\leqslant C\|u\|^{\frac{1}{2}}_{L^{\frac{6}{5}}(\Omega)\cap L^p(\Omega)} \cdot \|f\|^{\frac{1}{2}}_{L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)} \\
&\leqslant C\|u\|^{\frac{1}{2}}_{L^{\frac{6}{5}}(\Omega)\cap L^p(\Omega)} \cdot \big[\|u\|_{L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)} + \|Au\|_{L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)}\big]^{\frac{1}{2}}.
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
&\|\nabla u\|_{L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)} \\
&\quad \leqslant \|\nabla u\|_{L^{\frac{6}{5}}(\Omega)\cap L^p(\Omega)} + \|\nabla u\|_{L^{\frac{6}{5}}(O)\cap L^p(O)} \\
&\quad \leqslant \|\nabla u\|_{L^{\frac{6}{5}}(\Omega)\cap L^p(\Omega)} + C|\omega_u| \\
&\quad \leqslant C\|u\|^{\frac{1}{2}}_{L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)} \cdot \big[\|u\|_{L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)} + \|Au\|_{L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)}\big]^{\frac{1}{2}} + C\|u\|_{L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)} \\
&\quad \leqslant C\|u\|^{\frac{1}{2}}_{L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)} \|Au\|^{\frac{1}{2}}_{L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)} + C\|u\|_{L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)}.
\end{aligned}
\tag{6.51}
$$

For any $u_0 \in H_1^{\frac{6}{5}} \cap H_1^p$, applying (6.51) to $e^{-tA}u_0$ yields

$$
\begin{aligned}
\big\|\nabla e^{-tA}u_0\big\|_{L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)} &\leqslant C\big\|e^{-tA}u_0\big\|^{\frac{1}{2}}_{L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)} \big\|Ae^{-tA}u_0\big\|^{\frac{1}{2}}_{L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)} \\
&\quad + C\big\|e^{-tA}u_0\big\|_{L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)}.
\end{aligned}
$$

Note that

$$
\big\|e^{-tA}u_0\big\|_{L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)} \leqslant M_1\|u_0\|_{L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)},
$$

and

$$
\big\|Ae^{-tA}u_0\big\|_{L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)} \leqslant \frac{M_1}{t}\|u_0\|_{L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)}.
$$

Hence,

$$
\big\|\nabla e^{-tA}u_0\big\|_{L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)} \leqslant C_1(p,\Omega)(1+t^{-\frac{1}{2}})\|u_0\|_{L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)}. \tag{6.52}
$$

Let $u(t) = e^{-tA}u_0$, and $u(t) = l_u(t) + \omega_u(t) \times y$ in O. When $2 \leqslant p < 3$, and $q \in [p,\infty]$, using the Sobolev embedding inequality, one gets that

$$\begin{aligned}
\|u(t)\|_{L^q(\mathbb{R}^3)} &\leqslant \|u(t)\|_{L^q(\Omega)} + \|u(t)\|_{L^q(O)} \\
&\leqslant C\|u(t)\|^{\theta}_{L^p(\Omega)} \cdot \|u(t)\|^{1-\theta}_{W^{2,p}(\Omega)} + C\big[|l_u(t)| + |\omega_u(t)|\big] \\
&\leqslant C\|u(t)\|^{\theta}_{L^p(\Omega)} \cdot \big[\|u(t)\|_{L^p(\Omega)} + \|Au(t)\|_{L^p(\Omega)}\big]^{1-\theta} + C\|u(t)\|_{L^p(O)} \\
&\leqslant C\|u(t)\|^{\theta}_{L^p(\mathbb{R}^3)} \cdot \big[\|u(t)\|_{L^p(\mathbb{R}^3)} + \|Au(t)\|_{L^p(\mathbb{R}^3)}\big]^{1-\theta} + C\|u(t)\|_{L^p(\mathbb{R}^3)} \\
&\leqslant CM_1\|u_0\|^{\theta}_{L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)} \cdot \big[\|u_0\|_{L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)} + t^{-1}\|u_0\|_{L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)}\big]^{1-\theta} \\
&\quad + CM_1\|u_0\|_{L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)} \\
&\leqslant C(p,q,\Omega)\big[1 + t^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{q})}\big]\|u_0\|_{L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)},
\end{aligned}$$

where θ satisfies $\frac{1}{q} = \frac{\theta}{p} + (\frac{1}{p} - \frac{2}{3})(1-\theta)$.

When $p = 3$, $q \in [p, \infty)$, or $p > 3$, $q \in [p, \infty]$, using the Sobolev embedding inequality, one has

$$\begin{aligned}
\|u(t)\|_{L^q(\mathbb{R}^3)} &\leqslant C\|u(t)\|^{\theta}_{L^p(\mathbb{R}^3)} \cdot \|\nabla u(t)\|^{1-\theta}_{L^p(\mathbb{R}^3)} \\
&\leqslant C(p,q,\Omega)\big(1 + t^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{q})}\big)\|u_0\|_{L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)},
\end{aligned}$$

where θ satisfies $\frac{1}{q} = \frac{\theta}{p} + (\frac{1}{p} - \frac{1}{3})(1-\theta)$.

Therefore, we completed the proof of Proposition 6.1. □

Remark 6.2. Comparing to the estimates of classical Stokes semigroup in [14], here we are not able to get the corresponding decay estimates of $\nabla e^{-tA}u_0$. In Section 6 we will see that Proposition 6.1 is the key estimate to guarantee the local existence of a strong solution. However, without decay estimates of $\nabla e^{-tA}u_0$, we cannot get any global strong solution even when the initial data is small.

Remark 6.3. When O is a ball in $\mathbb{R}^3$, applying Theorem 2.9 instead of Theorem 2.5, we can prove the corresponding result for the case $e^{-tA_{2\cap p}}$, $p \geqslant 6$.

7. Local existence of strong solutions

In this section, assume that O is the unit ball in $\mathbb{R}^3$. We shall study the local existence of strong solutions to the system (1.2).

The proof of Theorem 2.12 is in spirit similar to those given in [16]. In fact, it was proved in [22], the system (1.2) can be rewritten in the abstract form

$$\partial_t v + Av + \mathbb{P}(v \cdot \nabla v) - \mathbb{P}(l_v \cdot \nabla v) = 0,$$

with the initial data

$$v(y,0) = v_0(y) = \begin{cases} a(y), & y \in \Omega, \\ b + c \times y, & y \in O. \end{cases}$$

Here $\mathbb{P}$ is the projection operator mentioned in Theorem 2.2, and l_v is associated with v such that $v = l_v + \omega_v \times y$ in O.

The above equation can be converted into the integral equation

$$v(y,t) = e^{-tA} v_0 - \int_0^t e^{-(t-s)A} \big[\mathbb{P}(\widehat{v \cdot \nabla v}) - \mathbb{P}(\widehat{l_v \cdot \nabla v})\big](s)\, ds,$$

where $\hat{f}$ denotes the restriction of f on the domain Ω, i.e.,

$$\hat{f}(y) = \begin{cases} f(y), & y \in \Omega, \\ 0, & y \in O. \end{cases}$$

Suppose that $\|v_0\|_{L^{\frac{6}{5}}(\mathbb{R}^3) \cap L^p(\mathbb{R}^3)} = K$, and set

$$X_{T_0} = \left\{ u(y,t) \;\middle|\; \begin{array}{l} \operatorname{div} u = 0 \text{ in } \mathbb{R}^3,\ D(u) = 0 \text{ in } O,\ \|u\|_{L^\infty(0,T_0; L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3))} \leqslant NK, \\ \text{and } \|t^{\frac{1}{2}} \nabla u\|_{L^\infty(0,T_0; L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3))} \leqslant NK \end{array} \right\}$$

where $N \geqslant 4 \max\{M_1, C_1\}$ and T_0 is to be determined later. Let

$$\|u\|_{X_{T_0}} = \max\big\{ \big\|u(t)\big\|_{L^\infty(0,T_0; L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3))}, \big\|t^{\frac{1}{2}} \nabla u(t)\big\|_{L^\infty(0,T_0; L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3))} \big\}.$$

For any $u \in X_{T_0}$, and $u = l_u + \omega_u \times y$ in O, define the map $\mathcal{L}$,

$$\mathcal{L}u = e^{-tA} v_0 - \int_0^t e^{-(t-s)A} \big[\mathbb{P}(\widehat{u \cdot \nabla u}) - \mathbb{P}(\widehat{l_u \cdot \nabla u})\big](s)\, ds.$$

We will show that, for proper T_0, $\mathcal{L}$ maps X_{T_0} int X_{T_0} and $\mathcal{L}$ is a contraction mapping.

$\mathcal{L}u$ can be estimated as the sum of three parts. Thanks to the estimates (2.7) and (6.49),

$$\big\|e^{-tA} v_0\big\|_{L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)} \leqslant M_1 K,$$

and

$$\big\|\nabla e^{-tA} v_0\big\|_{L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)} \leqslant C_1\big(1 + t^{-\frac{1}{2}}\big) K.$$

Since $\mathbb{P}$ is a bounded operator from $L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)$ to $H_1^{\frac{6}{5}}\cap H_1^p$, then it follows from the definition of X_{T_0} and Sobolev's inequality that

$$\begin{aligned}
&\left\|\int_0^t e^{-(t-s)A}\mathbb{P}(\widehat{u\cdot\nabla u})(s)\,ds\right\|_{L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)} \\
&\quad\leqslant \int_0^t M_1\big\|\mathbb{P}(\widehat{u\cdot\nabla u})(s)\big\|_{L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)}\,ds \\
&\quad\leqslant \int_0^t CM_1\big\|u(s)\big\|_{L^\infty(\mathbb{R}^3)}\cdot\big\|\nabla u(s)\big\|_{L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)}\,ds \\
&\quad\leqslant \int_0^t CM_1 s^{-\frac{3}{2p}}(NK)^2 s^{-\frac{1}{2}}\,ds \\
&\quad\leqslant \frac{2p}{p-3}CM_1(NK)^2 t^{\frac{1}{2}-\frac{3}{2p}} \\
&\quad= C_3(NK)^2 t^{\frac{1}{2}-\frac{3}{2p}},
\end{aligned}$$

where $C_3=C_3(p,\Omega)$ depends only on p and Ω.

$$\begin{aligned}
&\left\|\nabla\int_0^t e^{-(t-s)A}\mathbb{P}(\widehat{u\cdot\nabla u})(s)\,ds\right\|_{L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)} \\
&\quad\leqslant \int_0^t \big\|\nabla e^{-(t-s)A}\mathbb{P}(\widehat{u\cdot\nabla u})(s)\big\|_{L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)}\,ds \\
&\quad\leqslant \int_0^t CC_1\big[1+(t-s)^{-\frac{1}{2}}\big]\big\|(u\cdot\nabla u)(s)\big\|_{L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)}\,ds \\
&\quad\leqslant \int_0^t C\big[1+(t-s)^{-\frac{1}{2}}\big]\big\|u(s)\big\|_{L^\infty(\mathbb{R}^3)}\cdot\big\|\nabla u(s)\big\|_{L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)}\,ds \\
&\quad\leqslant \int_0^t C\big[1+(t-s)^{-\frac{1}{2}}\big]s^{-\frac{1}{2}}s^{-\frac{3}{2p}}(NK)^2\,ds \\
&\quad\leqslant C_4(p,\Omega)(NK)^2\big[t^{\frac{1}{2}-\frac{3}{2p}}+t^{-\frac{3}{2p}}\big].
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\left\| \int_0^t e^{-(t-s)A} \mathbb{P}(\widehat{l_u \cdot \nabla u})(s)\, ds \right\|_{L^{\frac{6}{5}}(\mathbb{R}^3) \cap L^p(\mathbb{R}^3)} \\
&\quad \leqslant \int_0^t C M_1 \left| l_u(s) \right| \cdot \left\| \nabla u(s) \right\|_{L^{\frac{6}{5}}(\mathbb{R}^3) \cap L^p(\mathbb{R}^3)}\, ds \\
&\quad \leqslant \int_0^t C M_1 \left\| u(s) \right\|_{L^\infty(\mathbb{R}^3)} \cdot \left\| \nabla u(s) \right\|_{L^{\frac{6}{5}}(\mathbb{R}^3) \cap L^p(\mathbb{R}^3)}\, ds \\
&\quad \leqslant \int_0^t C M_1 s^{-\frac{3}{2p}} (NK)^2 s^{-\frac{1}{2}}\, ds \\
&\quad \leqslant \frac{2p}{p-3} C M_1 (NK)^2 t^{\frac{1}{2} - \frac{3}{2p}} \\
&\quad = C_5(p, \Omega)(NK)^2 t^{\frac{1}{2} - \frac{3}{2p}}, \\
&\left\| \nabla \int_0^t e^{-(t-s)A} \mathbb{P}(\widehat{l_u \cdot \nabla u})(s)\, ds \right\|_{L^{\frac{6}{5}}(\mathbb{R}^3) \cap L^p(\mathbb{R}^3)} \\
&\quad \leqslant \int_0^t \left\| \nabla e^{-(t-s)A} \mathbb{P}(\widehat{l_u \cdot \nabla u})(s) \right\|_{L^{\frac{6}{5}}(\mathbb{R}^3) \cap L^p(\mathbb{R}^3)}\, ds \\
&\quad \leqslant \int_0^t C C_1 \left[1 + (t-s)^{-\frac{1}{2}}\right] \left\| (l_u \cdot \nabla u)(s) \right\|_{L^{\frac{6}{5}}(\mathbb{R}^3) \cap L^p(\mathbb{R}^3)}\, ds \\
&\quad \leqslant \int_0^t C \left[1 + (t-s)^{-\frac{1}{2}}\right] \left\| u(s) \right\|_{L^\infty(\mathbb{R}^3)} \cdot \left\| \nabla u(s) \right\|_{L^{\frac{6}{5}}(\mathbb{R}^3) \cap L^p(\mathbb{R}^3)}\, ds \\
&\quad \leqslant \int_0^t C \left[1 + (t-s)^{-\frac{1}{2}}\right] s^{-\frac{1}{2}} s^{-\frac{3}{2p}} (NK)^2\, ds \\
&\quad \leqslant C_6(p, \Omega)(NK)^2 \left[t^{-\frac{3}{2p}} + t^{\frac{1}{2} - \frac{3}{2p}}\right].
\end{aligned}$$

Hence

$$\left\| \mathcal{L}u(t) \right\|_{L^{\frac{6}{5}}(\mathbb{R}^3) \cap L^p(\mathbb{R}^3)} \leqslant M_1 K + (C_3 + C_5)(NK)^2 t^{\frac{1}{2} - \frac{3}{2p}},$$

and

$$\left\|t^{\frac{1}{2}}\nabla\mathcal{L}u(t)\right\|_{L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)} \leqslant C_1\left(1+t^{\frac{1}{2}}\right)K+(C_4+C_6)(NK)^2\left[t^{1-\frac{3}{2p}}+t^{\frac{1}{2}-\frac{3}{2p}}\right].$$

If T_0 is chosen to be sufficiently small such that

$$T_0 \leqslant T_1=\min\left\{\left[(C_3+C_5)NK\right]^{-\frac{2p}{p-3}},\left[(C_4+C_6)NK\right]^{-\frac{2p}{p-3}},1\right\},$$

then $\mathcal{L}$ maps X_{T_0} to X_{T_0}.

Furthermore, for any $u,\tilde{u}\in X_{T_0}$,

$$\begin{aligned}\mathcal{L}u-\mathcal{L}\tilde{u}={}&-\int_0^t e^{-(t-s)A}\left[\mathbb{P}(\widehat{u\cdot\nabla u})-\mathbb{P}(\widehat{l_u\cdot\nabla u})\right](s)\,ds\\&+\int_0^t e^{-(t-s)A}\left[\mathbb{P}(\widehat{\tilde{u}\cdot\nabla\tilde{u}})-\mathbb{P}(\widehat{l_{\tilde{u}}\cdot\nabla\tilde{u}})\right](s)\,ds\\={}&\int_0^t e^{-(t-s)A}\mathbb{P}\left[(\widehat{\tilde{u}-u\cdot\nabla)\tilde{u}}\right](s)\,ds+\int_0^t e^{-(t-s)A}\mathbb{P}\left[(\widehat{u\cdot\nabla)(\tilde{u}-u)}\right](s)\,ds\\&+\int_0^t e^{-(t-s)A}\mathbb{P}\left[(\widehat{l_u-l_{\tilde{u}}\cdot\nabla)u}\right](s)\,ds+\int_0^t e^{-(t-s)A}\mathbb{P}\left[(\widehat{l_{\tilde{u}}\cdot\nabla)(u-\tilde{u})}\right](s)\,ds.\end{aligned}$$

For each term on the right-hand side, the following estimates hold.

$$\begin{aligned}&\left\|\int_0^t e^{-(t-s)A}\mathbb{P}\left[(\widehat{\tilde{u}-u\cdot\nabla)\tilde{u}}\right](s)\,ds\right\|_{L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)}\\&\quad\leqslant\int_0^t CM_1\left\|\tilde{u}(s)-u(s)\right\|_{L^\infty(\mathbb{R}^3)}\cdot\left\|\nabla\tilde{u}(s)\right\|_{L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)}\,ds\\&\quad\leqslant\int_0^t CM_1 s^{-\frac{1}{2}}(NK)s^{-\frac{3}{2p}}\,ds\cdot\|u-\tilde{u}\|_{X_{T_0}}\\&\quad\leqslant C_7(p,\Omega)(NK)t^{\frac{1}{2}-\frac{3}{2p}}\|u-\tilde{u}\|_{X_{T_0}}.\end{aligned}$$

Similarly,

$$\left\|\int_0^t e^{-(t-s)A}\mathbb{P}\left[u\cdot\widehat{\nabla(\tilde{u}-u)}\right](s)\,ds\right\|_{L^{\frac{6}{5}}(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)}\leqslant C_8(p,\Omega)(NK)t^{\frac{1}{2}-\frac{3}{2p}}\|u-\tilde{u}\|_{X_{T_0}}.$$

Note that

$$|l_u - l_{\tilde{u}}| = \left| \frac{1}{m} \int_O (u - \tilde{u})\, dy \right| \leqslant C \|u - \tilde{u}\|_{L^\infty(\mathbb{R}^3)},$$

then

$$\left\| \int_0^t e^{-(t-s)A} \mathbb{P}\big[(l_u \widehat{- l_{\tilde{u}} \cdot} \nabla) u\big](s)\, ds \right\|_{L^{\frac{6}{5}}(\mathbb{R}^3) \cap L^p(\mathbb{R}^3)} \leqslant C_9(p, \Omega)(NK) t^{\frac{1}{2} - \frac{3}{2p}} \|u - \tilde{u}\|_{X_{T_0}},$$

and

$$\left\| \int_0^t e^{-(t-s)A} \mathbb{P}\big[(l_{\tilde{u}} \cdot \widehat{\nabla)(u -} \tilde{u})\big](s)\, ds \right\|_{L^{\frac{6}{5}}(\mathbb{R}^3) \cap L^p(\mathbb{R}^3)} \leqslant C_{10}(p, \Omega)(NK) t^{\frac{1}{2} - \frac{3}{2p}} \|u - \tilde{u}\|_{X_{T_0}}.$$

Hence, for every $t \in [0, T_0]$,

$$\big\| \mathcal{L} u(t) - \mathcal{L} \tilde{u}(t) \big\|_{L^{\frac{6}{5}}(\mathbb{R}^3) \cap L^p(\mathbb{R}^3)} \leqslant (C_7 + C_8 + C_9 + C_{10}) N K t^{\frac{1}{2} - \frac{3}{2p}} \|u - \tilde{u}\|_{X_{T_0}}.$$

Furthermore,

$$\begin{aligned}
&\left\| \nabla \int_0^t e^{-(t-s)A} \mathbb{P}\big[(\tilde{u} \widehat{- u \cdot} \nabla) \tilde{u}\big](s)\, ds \right\|_{L^{\frac{6}{5}}(\mathbb{R}^3) \cap L^p(\mathbb{R}^3)} \\
&\quad \leqslant \int_0^t C C_1 \big[1 + (t-s)^{-\frac{1}{2}}\big] \big\| \tilde{u}(s) - u(s) \big\|_{L^\infty(\mathbb{R}^3)} \cdot \big\| \nabla \tilde{u}(s) \big\|_{L^{\frac{6}{5}}(\mathbb{R}^3) \cap L^p(\mathbb{R}^3)}\, ds \\
&\quad \leqslant \int_0^t C C_1 \big[1 + (t-s)^{-\frac{1}{2}}\big] s^{-\frac{1}{2}} (NK) s^{-\frac{3}{2p}}\, ds \cdot \|u - \tilde{u}\|_{X_{T_0}} \\
&\quad \leqslant C_{11}(p, \Omega)(NK) \big[t^{\frac{1}{2} - \frac{3}{2p}} + t^{-\frac{3}{2p}}\big] \|u - \tilde{u}\|_{X_{T_0}}.
\end{aligned}$$

Similarly,

$$\left\| \nabla \int_0^t e^{-(t-s)A} \mathbb{P}\big[u \cdot \widehat{\nabla(\tilde{u} -} u)\big](s)\, ds \right\|_{L^{\frac{6}{5}}(\mathbb{R}^3) \cap L^p(\mathbb{R}^3)} \leqslant C_{12}(p, \Omega)(NK) t^{-\frac{3}{2p}} \|u - \tilde{u}\|_{X_{T_0}},$$

$$\left\| \nabla \int_0^t e^{-(t-s)A} \mathbb{P}\big[(l_u \widehat{- l_{\tilde{u}} \cdot} \nabla) u\big](s)\, ds \right\|_{L^{\frac{6}{5}}(\mathbb{R}^3) \cap L^p(\mathbb{R}^3)} \leqslant C_{13}(p, \Omega)(NK) t^{-\frac{3}{2p}} \|u - \tilde{u}\|_{X_{T_0}},$$

and

$$\left\| \nabla \int_0^t e^{-(t-s)A} \mathbb{P}\big[(l_{\tilde{u}} \cdot \widehat{\nabla)(u} - \tilde{u})\big](s)\, ds \right\|_{L^{\frac{6}{5}}(\mathbb{R}^3) \cap L^p(\mathbb{R}^3)}$$
$$\leqslant C_{14}(p, \Omega)(NK)\big[t^{\frac{1}{2} - \frac{3}{2p}} + t^{-\frac{3}{2p}}\big] \|u - \tilde{u}\|_{X_{T_0}}.$$

Let $T_2 = \min\{[(C_7 + C_8 + C_9 + C_{10})NK]^{-\frac{2p}{p-3}}, [(C_{11} + C_{12} + C_{13} + C_{14})NK]^{-\frac{2p}{p-3}}\}$. Combining the above estimates, one can obtain that when $T_0 \leqslant \min\{T_1, T_2\}$, $\mathcal{L}$ is a contraction mapping on X_{T_0}. Therefore, there exists a fixed point $v \in X_{T_0}$ of $\mathcal{L}$, i.e., $\mathcal{L}v = v$. It is clear that the fixed point $v(y, t) \in X_{T_0}$ is a strong solution to the system (1.2). The uniqueness of the solution is implied in the contraction property of $\mathcal{L}$.

Remark 7.1. Following almost the same proof of Theorem 2.12 and applying Theorem 2.9, we can also get a local strong solution starting from $H_1^2 \cap H_1^p$, $p \geqslant 6$.

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